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4.9 Antiderivatives (continued)

Example:

$$f''(x) = \frac{3}{\sqrt{x}}$$

$$f(1) = 0, f(4) = 1$$

Find $f(x)$.

$$f''(x) = 3x^{-1/2}$$

$$f'(x) = 3 \cdot \frac{1}{1/2} x^{1/2} + C = 6x^{1/2} + C$$

$$f(x) = 6 \cdot \frac{2}{3} x^{3/2} + Cx + D$$

$$f(x) = 4x^{3/2} + Cx + D$$

INITIAL CONDITIONS

$$f(1) = 0 \Rightarrow 4 + C + D = 0 \Rightarrow C + D = -4$$

$$f(4) = 1 \Rightarrow 4(4)^{3/2} + C(4) + D = 1 \quad D = -4 - C$$

$$\Rightarrow 32 + 4C + D = 1$$

$$\Rightarrow 4C + D = -31$$

$$4C + (-4 - C) = -31$$

$$3C - 4 = -31$$

$$3C = -27$$

$$C = -9$$

$$D = -4 - C = -4 - (-9) = 5$$

$$f(x) = 4x^{3/2} - 9x + 5$$

CHECK!!!

$$f(1) = 4 - 9 + 5 = 0 \quad \checkmark$$

$$f(4) = 4(4)^{3/2} - 9(4) + 5 = 32 - 36 + 5 = 1 \quad \checkmark$$

$$f'(x) = 6x^{1/2} - 9$$

$$f''(x) = 3x^{-1/2} \quad \checkmark$$

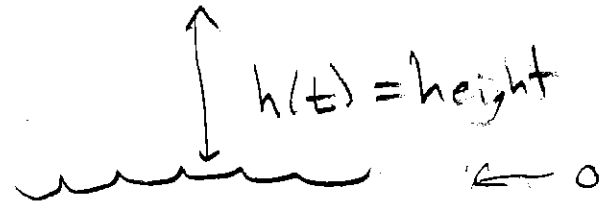
Example:

Ron steps off the 10 meter high dive at his local pool. Find a formula for his height above the water.

(Assume his acceleration is a constant 9.8 m/s^2 downward)



$$h(0) = 10$$
$$v(0) = 0$$



$$a(t) = -9.8$$

$$h''(t) = -9.8, \quad h(0) = 10$$
$$h'(0) = 0$$

$$\Rightarrow h'(t) = -9.8t + C$$

$$h(t) = -4.9t^2 + Ct + D$$

INITIAL CONDITIONS:

$$h'(0) = 0 \Rightarrow -9.8(0) + C = 0 \Rightarrow C = 0$$

$$h(0) = 10 \Rightarrow -4.9(0)^2 + C(0) + D = 10 \Rightarrow D = 10$$

$$h(t) = -4.9t^2 + 10$$

CHECK!

5.1 Defining Area (Riemann sums)

Calculus is based on limiting processes that “approach” the exact answer to a rate question.

In Calculus I, you defined

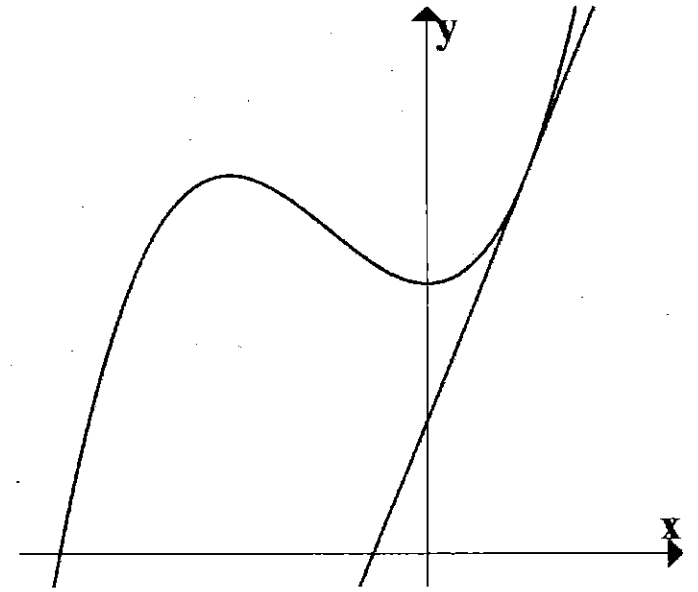
$f'(x)$ = `slope of the tangent at x '

$$= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

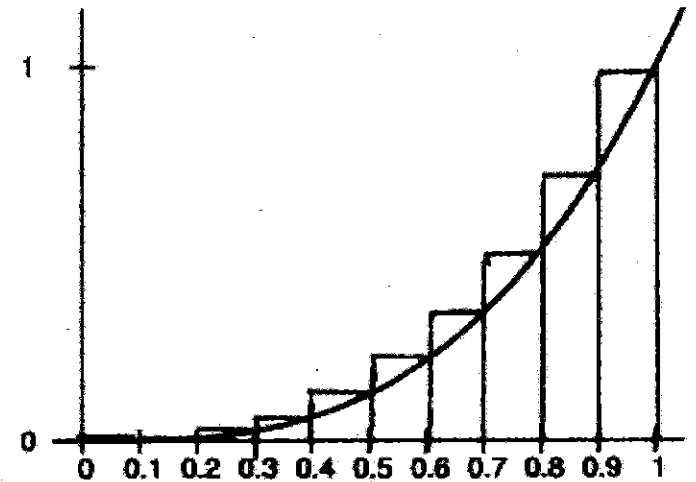
In Calculus II, we will see that antiderivatives are related to the area `under' a graph

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

Calc. I
Visual:



Calc. II
Visual:



$$R_{10} = 0.3025$$

Riemann sums set up:

We are going to build a procedure to get better and better approximations of the area "under" $f(x)$.

1. Break into n equal subintervals.

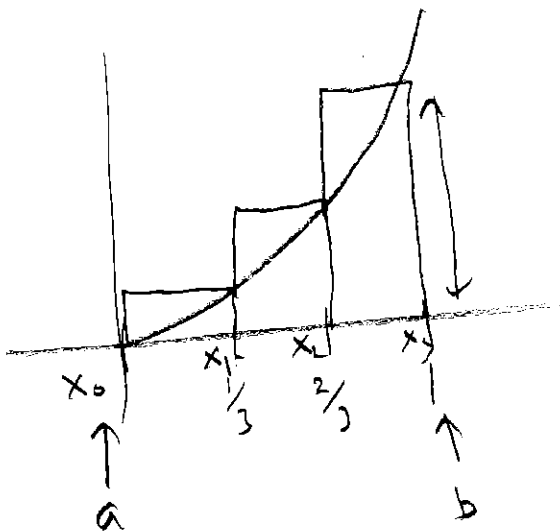
$$\Delta x = \frac{b-a}{n} \text{ and } x_i = a + i\Delta x$$

2. Draw n rectangles; use function.

Area of each rectangle =

$$(\text{height})(\text{width}) = f(x_i^*)\Delta x$$

3. Add up rectangle areas.



Example:

Approximate the area under $f(x) = x^3$ from $x = 0$ to $x = 1$ using $n = 3$ subdivisions and *right-endpoints* to find the height.

$$a = 0 \\ b = 1$$

$$f(x) = x^3$$

$$n = 3$$

$$\boxed{\text{I}} \quad \Delta x = \frac{b-a}{n} = \frac{1-0}{3} = \frac{1}{3}$$

$$x_0 = a = 0$$

$$x_1 = a + \Delta x = 0 + \frac{1}{3} = \frac{1}{3}$$

$$x_2 = a + 2\Delta x = 0 + 2 \cdot \frac{1}{3} = \frac{2}{3}$$

$$x_3 = a + 3\Delta x = 0 + 3 \cdot \frac{1}{3} = 1$$

$\boxed{\text{II}}$

$$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x$$

$$f\left(\frac{1}{3}\right)\frac{1}{3} + f\left(\frac{2}{3}\right)\frac{1}{3} + f(1)\frac{1}{3}$$

$$\left(\frac{1}{3}\right)^3 \frac{1}{3} + \left(\frac{2}{3}\right)^3 \frac{1}{3} + 1^3 \frac{1}{3}$$

$$\approx 0.493827 = R_3$$

OVERESTIMATE!

Closing Wed: HW_1A, 1B, 1C

Entry Task (You do): Approx. the area under $f(x) = x^3$ from $x = 0$ to $x = 1$ using $n = 4$ and *right-endpoints*.

Step 1: $\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$

Step 2: $x_0 = a = 0$
 $x_1 = a + \Delta x = 0 + \frac{1}{4}$
 $x_2 = a + 2\Delta x = 0 + 2(\frac{1}{4})$
 $x_3 = a + 3\Delta x = 0 + 3(\frac{1}{4})$
 $x_4 = a + 4\Delta x = 0 + 4(\frac{1}{4})$

Step 3: Plug in right-endpoints to function to get rect. heights, then add up areas (height times width).

$$\text{Area} \approx \sum_{i=1}^4 f(x_i)\Delta x =$$

$$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$

$$\left(\frac{1}{4}\right)^3 \frac{1}{4} + \left(\frac{2}{4}\right)^3 \frac{1}{4} + \left(\frac{3}{4}\right)^3 \frac{1}{4} + \left(\frac{4}{4}\right)^3 \frac{1}{4}$$

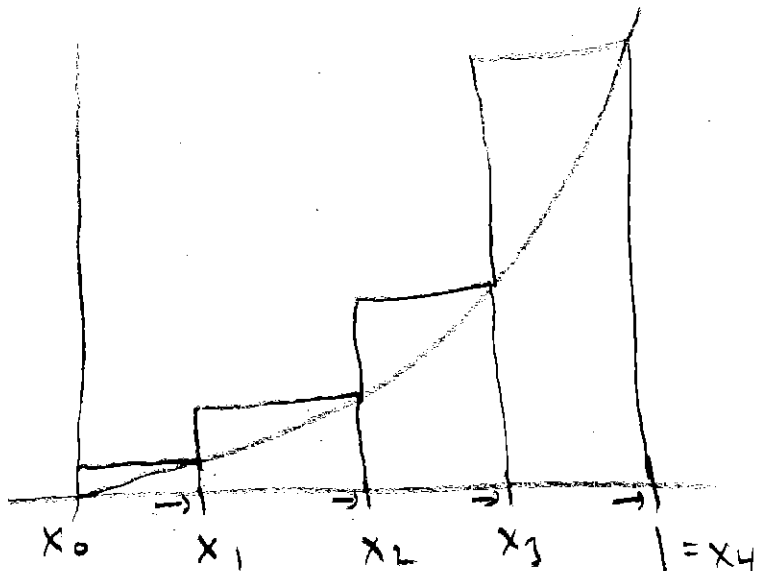
Right-endpoints

= 0.390625

$$\left(\frac{1}{4}\right)^3 \frac{1}{4} + \left(\frac{2}{4}\right)^3 \frac{1}{4} + \left(\frac{3}{4}\right)^3 \frac{1}{4} + \left(\frac{4}{4}\right)^3 \frac{1}{4}$$

PATTERN $\sum_{i=1}^4 \left(\frac{i}{4}\right)^3 \frac{1}{4} = \frac{1}{4^4} \sum_{i=1}^4 i^3$

ASIDE



You do:

Approx. the area under $f(x) = x^3$
from $x = 0$ to $x = 1$ using $n = 4$
subdivisions and *left-endpoints* to
find the heights.

did this example again with 100 subdivisions, then 1000, then 10000. Here is a summary of my findings:

n	R_n	L_n
4	0.390625	0.140625
5	0.36	0.16
10	0.3025	0.2025
100	0.255025	0.245025
1000	0.25050025	0.24950025
10000	0.2499500025	0.2500500025

Pattern:

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}, \quad x_i = 0 + i \frac{1}{n} = \frac{i}{n}$$

Adding up the area of each rectangle

$$\text{Sum} = \sum_{i=1}^n x_i^3 \Delta x = \sum_{i=1}^n \left(\frac{i}{n}\right)^3 \frac{1}{n}$$

$$\text{Area} = 0.25 = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{i}{n}\right)^3 \frac{1}{n}$$

$\lim_{n \rightarrow \infty} \left(\frac{1}{n^4} \sum_{i=1}^n i^3 \right)$
 ASIDE

Example: Approximate the area under $f(x) = 1 + x^2$ from $x = 2$ to $x = 3$ using Riemann sums with $n = 4$ and right endpoints.

$$\Delta x = \frac{3-2}{4} = \frac{1}{4}$$

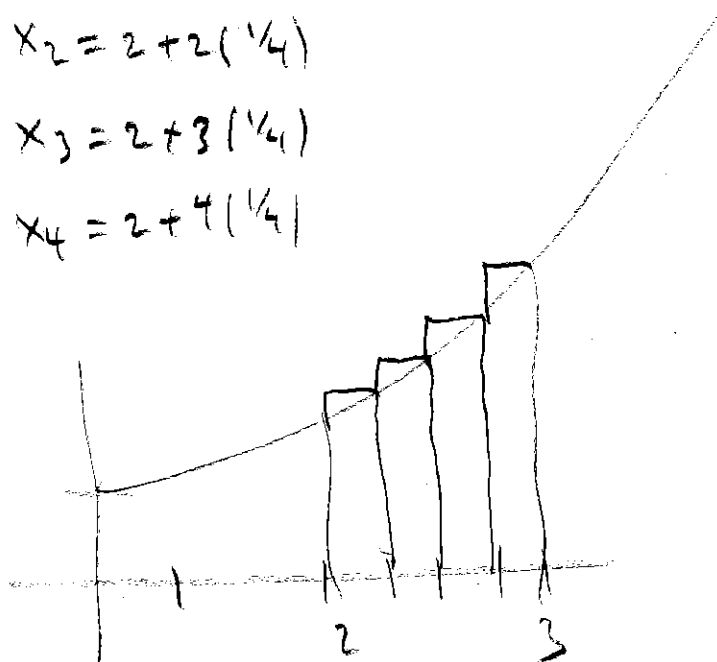
$$x_0 = 2$$

$$x_1 = 2 + \frac{1}{4}$$

$$x_2 = 2 + 2\left(\frac{1}{4}\right)$$

$$x_3 = 2 + 3\left(\frac{1}{4}\right)$$

$$x_4 = 2 + 4\left(\frac{1}{4}\right)$$



$$(1 + (2.25)^2) \frac{1}{4} + (1 + (2.5)^2) \frac{1}{4} + (1 + (2.75)^2) \frac{1}{4} + (1 + (3)^2) \frac{1}{4} = 7.96875$$

What is the general pattern in terms of n ?

$$\Delta x = \frac{3-2}{n} = \frac{1}{n}$$

$$x_i = 2 + i \frac{1}{n} = 2 + \frac{i}{n}$$

$$\sum_{i=1}^n f(x_i) \Delta x = \sum_{i=1}^n (1 + x_i^2) \Delta x$$

$$= \sum_{i=1}^n \left(1 + \left(2 + \frac{i}{n} \right)^2 \right) \frac{1}{n}$$

\uparrow a \leftarrow $b-a$

Another Example:

Using sigma notation, write down the general Riemann sum definition of the area from $x = 5$ to $x = 7$ under

$$f(x) = 3x + \sqrt{x}$$

$$\Delta x = \frac{b-a}{n} = \frac{7-5}{n} = \frac{2}{n}$$

$$x_i = a + i \Delta x = 5 + i \left(\frac{2}{n}\right) = 5 + \frac{2i}{n}$$

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n (3x_i + \sqrt{x_i}) \Delta x$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(3\left(5 + \frac{2i}{n}\right) + \sqrt{5 + \frac{2i}{n}} \right) \frac{2}{n}$$

$a=5$ $b-a=2$

Velocity/Distance & Reimann Sums

When velocity is a **constant**:

$$\text{Distance} = \text{Velocity} \cdot \text{Time}$$

Example:

You are accelerating in a car. You get the following measurements:

t (sec)	0	0.5	1.0	1.5	2.0
v(t) (ft/s)	0	6.2	10.8	14.9	18.1

Estimate the distance traveled by the car traveled from 0 to 2 seconds.

HAVE TO BREAK IT UP!!

	LOW ESTIMATE	HI ESTIMATE
0 to 0.5	$0 \frac{\text{ft}}{\text{sec}} \cdot 0.5 \text{ sec} = 0 \text{ ft}$	$6.2 \frac{\text{ft}}{\text{sec}} \cdot 0.5 \text{ sec} = 3.1 \text{ ft}$
0.5 to 1	$6.2 \frac{\text{ft}}{\text{sec}} \cdot 0.5 \text{ sec} = 3.1 \text{ ft}$	$10.8 \frac{\text{ft}}{\text{sec}} \cdot 0.5 \text{ sec} = 5.4 \text{ ft}$
1 to 1.5	$10.8 \frac{\text{ft}}{\text{sec}} \cdot 0.5 \text{ sec} = 5.4 \text{ ft}$	$14.9 \frac{\text{ft}}{\text{sec}} \cdot 0.5 \text{ sec} = 7.45 \text{ ft}$
1.5 to 2	$14.9 \frac{\text{ft}}{\text{sec}} \cdot 0.5 \text{ sec} = 7.45 \text{ ft}$	$18.1 \frac{\text{ft}}{\text{sec}} \cdot 0.5 \text{ sec} = 9.05 \text{ ft}$
TOTAL =	15.95 ft	25 ft

ASIDE: $\text{UNITS} = \frac{\text{ft}}{\text{sec}} \cdot \text{sec} = \text{ft}$

\nearrow "HEIGHT" UNITS "WIDTH" UNITS

5.2 The Definite Integral

Def'n:

We define the **definite integral of $f(x)$ from $x = a$ to $x = b$** by

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x, \quad \leftarrow$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

NOTES:

" \int " = integral sign

a, b = bounds or limits of integration

$\int_a^b f(x) dx =$ "ADD UP" $f(x_i) \Delta x$
ACROSS THE INTERVAL

= A NUMBER:

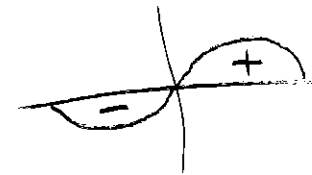


$$\int_0^3 7 dx = 21$$

$$\int_0^3 -5 dx = -15$$

"SIGNED" or "NET" AREA

$$\int_{-\pi}^{\pi} \sin(x) dx = 0$$



Basic Integral Rules:

$$1. \int_a^b c \, dx = (b - a)c$$

$$2. \int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx$$

$$3. \int_a^b cf(x) \, dx = c \int_a^b f(x) \, dx$$

and

$$\int_a^b f(x) + g(x) \, dx$$

$$= \int_a^b f(x) \, dx + \int_a^b g(x) \, dx$$

$$4. \int_b^a f(x) \, dx = - \int_a^b f(x) \, dx$$

Examples:

$$1. \int_4^{10} 5 \, dx = 5(10 - 4) = 30$$

$$2. \int_0^3 x^2 \, dx + \int_3^7 x^2 \, dx = \int_0^7 x^2 \, dx$$

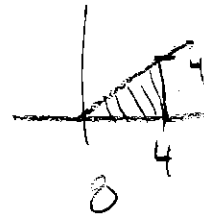
$$3. \int_0^4 5x + 3 \, dx = \int_0^4 5x \, dx + \int_0^4 3 \, dx$$

$$= 5 \int_0^4 x \, dx + \int_0^4 3 \, dx$$

$$= 5 \cdot 8$$

$$+ 12$$

$$= \boxed{52}$$



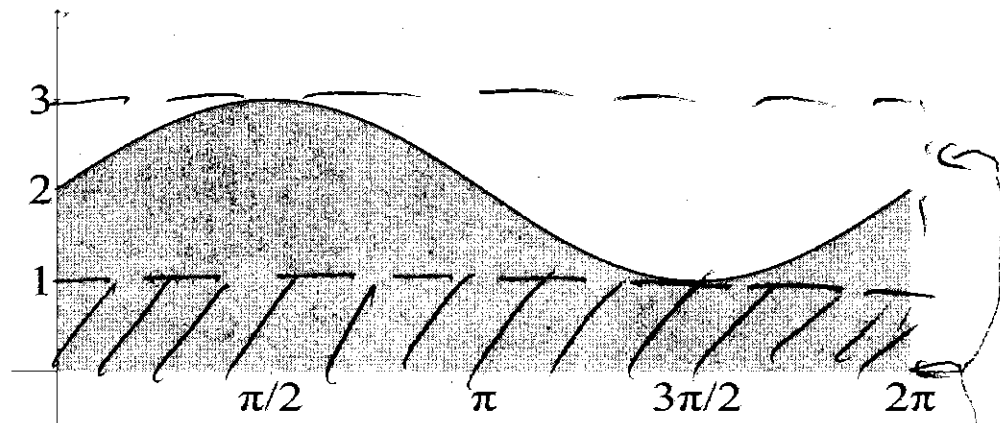
$$4. \int_3^1 x^3 \, dx = - \int_1^3 x^3 \, dx$$

Note on quick bounds (HW_1C: 9,10)

$$m(b - a) \leq \int_a^b f(x) dx \leq M(b - a)$$

Example: Consider the area under $f(x) = \sin(x) + 2$ on the interval $x = 0$ to $x = 2\pi$.

- (a) What is the max of $f(x)$? (label M)
- (b) What is the min of $f(x)$? (label m)
- (c) Draw **one** rectangle that contains all the shaded area? What can you conclude?
- (d) Draw **one** rectangle that is completely inside the shaded area? Conclusion?



$$m \leq \sin(x) + 2 \leq M$$

SO $\int_0^{2\pi} \sin(x) + 2 dx$

MUST BE BETWEEN THE AREA OF THESE TWO RECTANGLES

$$\underbrace{1 \cdot 2\pi}_{2\pi} \leq \int_0^{2\pi} \sin(x) + 2 dx \leq \underbrace{3 \cdot 2\pi}_{6\pi}$$